TRUNCATED GAUSSIAN KRIGING AS AN ALTERNATIVE TO INDICATOR KRIGING

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ABSTRACT

Truncated Gaussian simulation (TGS) and plurigaussian simulation (PGS) are widely accepted methods for generating realisations of geological domains (lithofacies) that reproduce contact relationships. The realisations can be used to evaluate transfer functions related to the lithofacies occurrence, the simplest ones of which are the probability of occurrence of each lithofacies and the most probable lithofacies at each location of the deposit.

In order to get the probability of occurrence of a lithofacies, the simulation approach can be time consuming. A shortcut method (truncated Gaussian kriging, or TGK) is proposed, based on the truncated Gaussian simulation model and the well-known multi-Gaussian kriging method. In this method, the variogram analysis stage and the definition of the truncation rule remain the same as in the traditional truncated Gaussian simulation approach.

The formulation of the method is halfway between spatial estimation and simulation. The key point is to apply the truncation rule to the local distribution of the underlying Gaussian random field used in the TGS approach. Because the relationship between the lithofacies indicators and this Gaussian random field is not one-to-one, the latter is simulated at the data locations conditionally to the available indicator data. The local distributions of the Gaussian random field at the target locations are then obtained by considering the simple kriging estimates and simple kriging variances, as it is done in the multi-Gaussian kriging approach.

TGK can be used as a step previous to simulating lithofacies, or as an alternative to indicator kriging, when the lithofacies exhibit a hierarchical spatial disposition or when such a disposition is a desirable feature. The proposed method is naturally extensible to plurigaussian simulation.

INTRODUCTION

Currently, numerical models of the spatial distribution of geological domains (*lithofacies*) can be generated by geostatistical methods. Two approaches are commonly used to achieve this goal: stochastic simulation and local uncertainty models.

Stochastic simulation consists of creating multiple realisations of the lithofacies of the deposit. The realisations can be used to evaluate transfer functions related to the lithofacies occurrence, the simplest of which is the probability of occurrence of each lithofacies. In contrast, local uncertainty models directly provide the probabilities of occurrence of the lithofacies without generating several realisations. The main methods associated with stochastic simulation are truncated Gaussian (TGS), plurigaussian (PGS) and sequential indicator (SIS) simulation, whereas the most common method for local uncertainty models is indicator kriging (IK). This paper presents a local-stochastic approach to obtain the probability of occurrence of each lithofacies based on truncated Gaussian simulation and multi-Gaussian kriging (MGK).

OVERVIEW OF CURRENT METHODS

Indicator Kriging (IK)

Indicator kriging [1, 2] is a non-parametric technique to calculate the conditional cumulative distribution function (CCDF) of a set of indicators, which are a binary coding of a categorical variable representing the lithofacies. It basically consists of estimating the indicator values using a kriging or cokriging of indicator data. The estimated values of each indicator are interpreted as the probability density function of the lithofacies, generating a local model of the probabilities of occurrence of lithofacies.

Sequential Indicator Simulation (SIS)

Sequential indicator simulation [2] rests on the sequential estimation of the CCDF associated with the lithofacies coded as indicators. The estimation is performed by indicator kriging using the sample data and previously simulated nodes as conditioning information. From the CCDF a lithofacies is drawn by Monte Carlo simulation at each node. The main advantages of the method are: its auto-conditional nature, the simple incorporation of soft data and the possibility to express spatially highly continuous patterns. As a counterpart, IK and consequently SIS suffer from order relation violations in the CCDF, among others problems [3].

Truncated Gaussian Simulation (TGS)

The truncated Gaussian simulation method [4] relies on the truncation of a single Gaussian random field (GRF) in order to generate realisations of lithofacies. The main feature is the reproduction of the indicator variograms associated with the lithofacies and the hierarchical contact relationship among them. This method is adequate for deposits where the lithofacies exhibit a hierarchical spatial distribution, such as depositional environments or sedimentary formations.

The procedure to obtain lithofacies realisations using TGS is described as follows:

- Establish the lithofacies proportions and their contact relationships. Summarise this information in a truncation rule (flag).
- Using the truncation rule, perform variography of the lithofacies indicators through the determination of the covariance function of the underlying GRF.
- Simulate the GRF at the data locations conditionally to the lithofacies coding. This step is performed using the Gibbs sampler algorithm [5]. As the relationship between the lithofacies indicators and GRF is not one-to-one, several realisations should be considered for the next steps.

- Simulate the GRF at the target locations using the values generated at the previous step as conditioning data.
- Truncate the realisations according to the truncation rule.

Plurigaussian Simulation (PGS)

Plurigaussian simulation [6, 7] is an extension of truncated Gaussian simulation that incorporates two or more Gaussian random fields and a set of truncation rules. The use of several GRFs allows reproducing complex contact relationships between the lithofacies. The workflow of PGS is similar to TGS.

Multi-Gaussian Kriging (MGK)

Multi-Gaussian kriging [8] is a method to calculate the conditional distribution of a GRF at a point support. It has been used to establish the risk of exceeding or falling short of a threshold for a continuous (not necessarily Gaussian) variable. It relies on the application of the multi-Gaussian hypothesis and the property of orthogonality of simple kriging.

The key property of the multi-Gaussian model is that the multivariate distributions of a GRF are fully defined by its first- and second-order moments: mean and covariance function. The orthogonality property is that the simple kriging estimator is not correlated with any linear combination of the data. Therefore, it can be shown that the conditional distribution of a GRF is Gaussian, with mean equal to the simple kriging estimate and variance equal to the simple kriging variance.

The workflow of the application of multi-Gaussian kriging to get the conditional distribution of a continuous variable is described below:

- Transform the raw variable into a Gaussian variable. Store the transformation table.
- Perform simple kriging of the Gaussian variable. At each target location, the conditional distribution is fully defined by the simple kriging estimate and simple kriging variance.
- Perform numerical integration at each target location:
  - Sample the conditional Gaussian distribution using Monte Carlo simulation
  - Back-transform every sampled Gaussian value according to the transformation table
  - The distribution of back-transformed values is an approximation to the distribution of the original variable conditional to the available data. From this distribution, several measures can be derived, e.g. expected value (mean of the distribution), conditional variance (variance of the distribution), probability to exceed a given threshold, or confidence intervals.

PROPOSED APPROACH: TRUNCATED GAUSSIAN KRIGING (TGK)

The proposed method is based on the following aspects to get the probability of occurrence of a lithofacies:

- To generate realisations of lithofacies, TGS use Gaussian simulations that rely on the multi-Gaussian hypothesis.
- The conditional distributions of the underlying GRF used in TGS can be obtained by the multi-Gaussian kriging approach.
- The truncation rule can be interpreted as a particular transformation from a categorical variable (lithofacies) to a continuous variable (GRF). This transformation is similar to

the one used to get the conditional distribution of a continuous variable in MGK, except that the truncation rule is not one-to-one.

Therefore, it is possible to calculate the probability of occurrence of each lithofacies without simulating the GRF in the domain. Instead the multi-Gaussian approach can be used to obtain the conditional distribution at each target location. As the truncation rule is not one-to-one, we will need several independent realisations of the GRF at the data locations as conditioning data (see TGS workflow). Therefore we will have to truncate several conditional Gaussian distributions with different mean values but with the same kriging variance; recall that the simple kriging variance does not depend on the data values.

The workflow of the proposed method is presented only for the stationary case, i.e., when the proportions of the lithofacies remain constant over the domain under study.

Consider $F_1 \dots F_n$ as n contiguous lithofacies present in the deposit. The indicators associated with these lithofacies are defined as:

$$1_i(x) = \begin{Bmatrix} 1\ if\ x\ \in F_i \\ 0\ otherwise \end{Bmatrix} \forall\ i = 1 \dots n, \forall\ x \in \mathbb{R}^3 \quad (1)$$

Let $Y(x)$ be a standard GRF with covariance function $C_Y(h)$. The n lithofacies are defined by $n-1$ thresholds and every lithofacies can be expressed as the truncation of $Y(x)$ as follows:

$$F_i = \{x \in \mathbb{R}^3, l^i \leq Y(x) < u^i\} \ \forall\ i = 1 \dots n \quad (2)$$

where l^i and u^i stand for the lower and upper truncation thresholds for the *i*-th lithofacies and $l^i = u^{i-1}\ \forall\ i = 2 \dots n-1$. For lithofacies F_1 and F_n the lower and upper thresholds are set to $l^1 = -\infty$ and $u^n = +\infty$, respectively. The proportion of the *i*-th lithofacies is defined by:

$$P_i = E\big(1_i(x)\big) = prob\big(l^i \leq Y(x) < u^i\big) = G\big(u^i\big) - G\big(l^i\big) \quad (3)$$

with G the standard Gaussian cumulative density function. Keeping this notation in mind, the workflow of TGK is the following:

- Establish the lithofacies proportions and their contact relationships. Summarise this information in a truncation rule.
- Using the truncation rule, perform variography of the lithofacies indicators through the determination of the covariance $C_Y(h)$ of the underlying GRF.
- Simulate the GRF at the data locations conditionally to the lithofacies coding. Several ($nrealis$) realisations or sets of Gaussian values are generated at this step.
- Perform simple kriging using the covariance of the GRF and the previous realisations as conditioning data. A single execution of simple kriging is needed to determine the kriging weights and kriging variance.
- At this stage, we have several kriging estimates and a single kriging variance at each target location. Using the multi-Gaussian hypothesis, the conditional probability for the *i*-th lithofacies and *j*-th realisation can be expressed as follows:

$$Prob(x \in F_i\,, realisation\ j) = G\left(\frac{u^i - Y_{SK}^j(x)}{\sigma_{SK}(x)}\right) - G\left(\frac{l^i - Y_{SK}^j(x)}{\sigma_{SK}(x)}\right) \quad (4)$$

where $Y_{SK}^j(x)$ is the simple kriging estimate calculated on realisation j and $\sigma_{SK}^2(x)$ is the simple kriging variance at location x.

- The final probability for each lithofacies at location x can be expressed as:

$$Prob(x \in F_i) = \frac{1}{nrealis} \sum_{j=1}^{nrealis} \left(G\left(\frac{u^i - Y_{SK}^j(x)}{\sigma_{SK}(x)}\right) - G\left(\frac{l^i - Y_{SK}^j(x)}{\sigma_{SK}(x)}\right) \right) \quad (5)$$

Because of the use of multiple realisations of the GRF at the data locations and of simple kriging to obtain the conditional distribution of the GRF at the target locations, the proposed method is in-between simulation and kriging estimation.

APPLICATION OF TRUNCATED GAUSSIAN KRIGING

A synthetic case study is presented, which considers the estimation of the probability of three lithofacies (coded as lith1, lith2 and lith3) that are embedded units. A set of 189 sample data is available to calculate the probabilities. The data are distributed over a 500 m × 500 m domain, as shown in Figure 1.

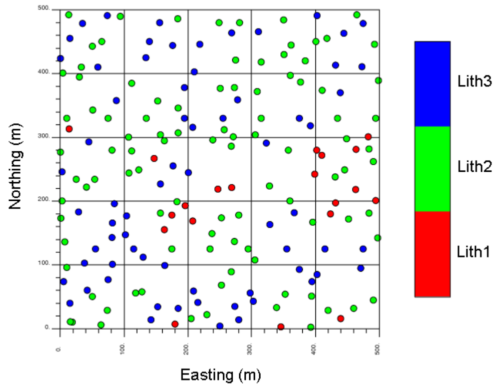

Figure 1: Data locations showing lithofacies coding

Basic parameters. The contact relationships, declustered proportions and threshold values are summarised in Figure 2. The truncation rule reflects the hierarchical disposition of the lithofacies.

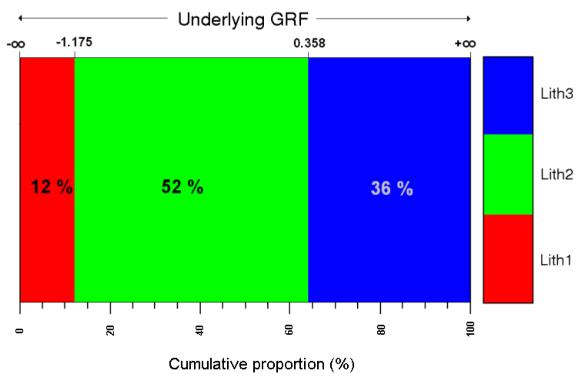

Figure 2: Truncation rule, showing contact relationship, proportions and Gaussian thresholds associated with the lithofacies

Variography. At this step the indicator variograms are fitted by defining the covariance $C_Y(h)$ of underlying GRF (Table 1).

Table 1: Covariance model of the underlying GRF

Structure	Sill contribution	Major range (60°E)	Minor range (30°W)
Gaussian	1	120	80

Gibbs sampler. Several sets of Gaussian values are generated at the data locations in order to honour the truncation rule and the covariance function $C_Y(h)$. Two realisations are presented in Figure 3.

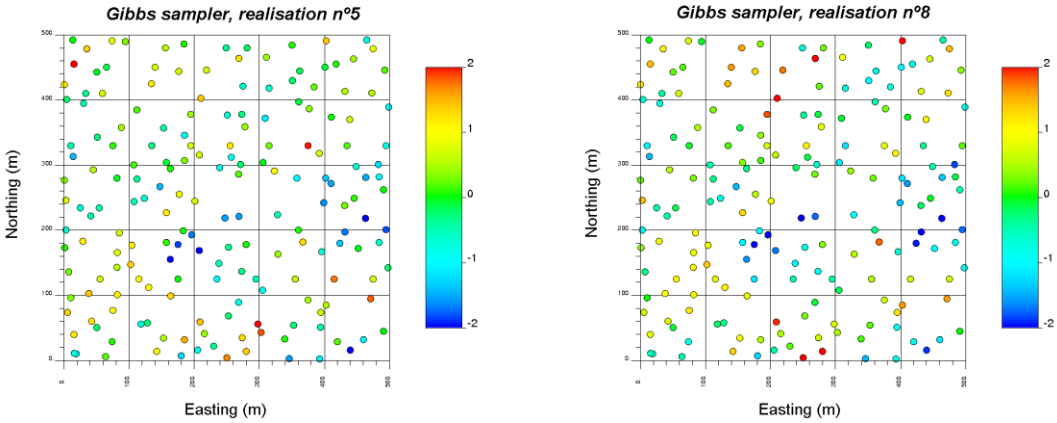

Figure 3: Two realisations of the Gibbs sampler algorithm

Modelling the local distributions. Simple kriging is performed, given the covariance $C_Y(h)$ and the realisations at the data locations as conditioning data. Figure 4 presents two simple kriging estimates, derived from the realisations shown in Figure 3 and their kriging variance, which fully defines the conditional distributions of the GRF.

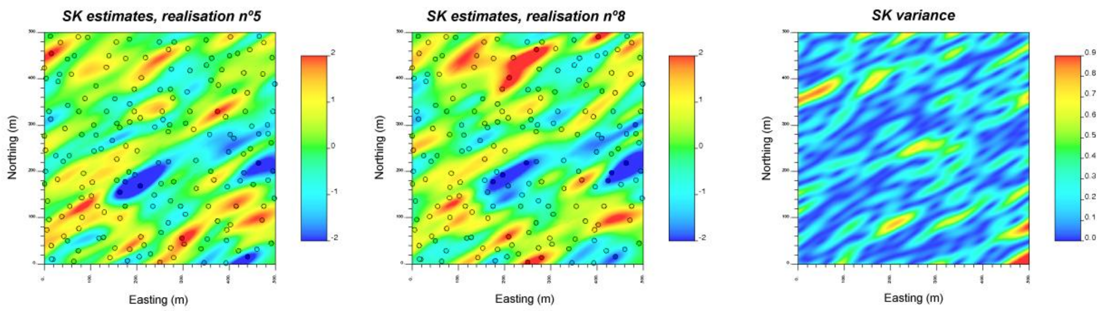

Figure 4: Two simple kriging estimates from two Gibbs sampler realisations and kriging variance

Calculation of the conditional probabilities of lithofacies. The truncation rule is applied to the local distributions of the underlying GRF (Eq. 5). The thresholds are directly computed by using the global proportions of lithofacies, as per Eq. 3.

Figure 5 presents a workflow of the procedure, where the upper Gaussian distribution represents the prior (non-conditional) model used with the contact relationship and global proportions expressed as the truncation of the underlying GRF by threshold T1 and T2. The lower Gaussian distribution represents a conditional distribution of the GRF at a given location obtained by MGK.

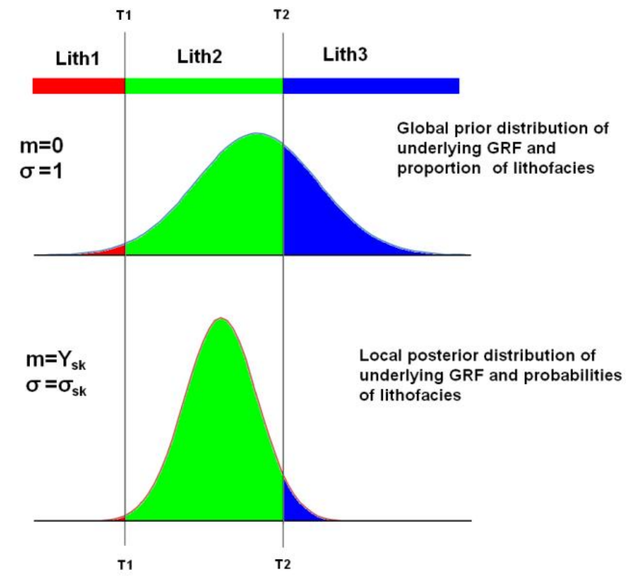

Figure 5: Local probability calculations

The resulting local probabilities and conditioning data are presented in Figure 6. The most probable lithofacies is also calculated and presented in Figure 7, where the contact relationship imposed by the truncation rule is clearly expressed.

For comparison, an indicator kriging of the lithofacies was performed using the same data and search parameters as in TGK. In this case the most probable lithofacies show violations of the contact relationship in several instances (Figure 8). This feature happens when there are data of lith1 near to lith3 without data of lith2 to restrict the estimation. At the same locations the truncated Gaussian kriging approach (Figure 7) generates the intermediate unit (lith2).

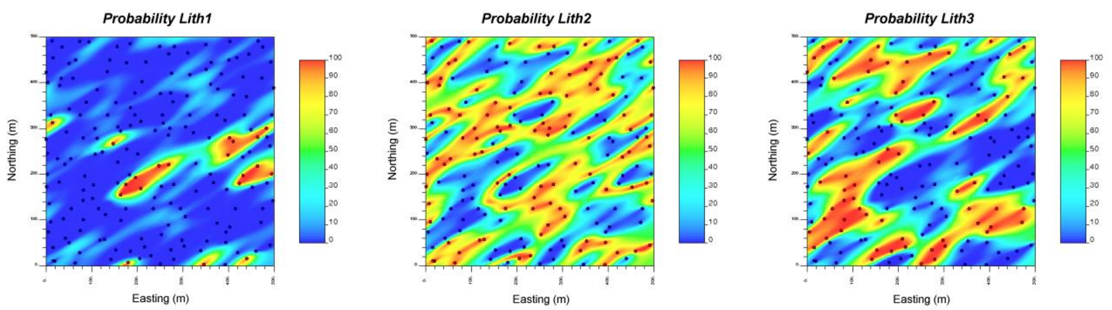

Figure 6: Maps of probabilities of occurrence of each lithofacies using TGK

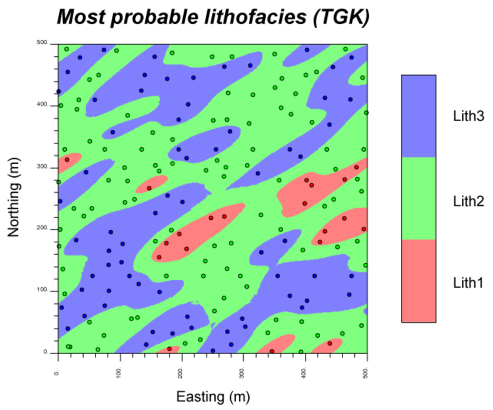

Figure 7: Most probable lithofacies using truncated Gaussian kriging (TGK)

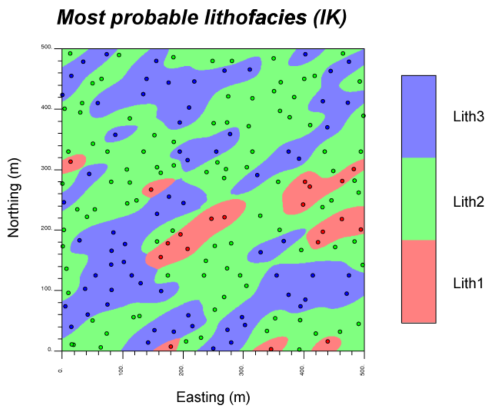

Figure 8: Most probable lithofacies using indicator kriging (IK)

DISCUSSION

The proposed approach allows generating a probability model of each lithofacies that presents a hierarchical disposition, whereas for indicator kriging this feature is not guaranteed and the amount of "violations" is likely to increase with the number of lithofacies.

TGK is naturally extensible to consider complex contact relationships between lithofacies, using the plurigaussian simulation framework instead of the truncated Gaussian. In this case the proposed method needs to determinate the conditional multivariate distribution of two or more underlying GRFs. To achieve this goal, it is necessary to perform multi-Gaussian kriging or cokriging, depending on whether or not the GRFs are correlated. The authors are working on that extension named *plurigaussian kriging*.

In geostatistics, there is a correspondence between some stochastic imaging methods and local uncertainty models, as described in Table 2. For sequential indicator simulation and Gaussian simulation, there already exists an associated model of local uncertainty. However for truncated Gaussian and plurigaussian simulations, there was no associated model until the present work.

Table 2: Correspondence between stochastic imaging and local uncertainty models

Local uncertainty model	Stochastic imaging	Type of model	Type of variable
Indicator kriging	Sequential indicator simulation	non-parametric	Categorical / Continuous
Multi-Gaussian kriging	Gaussian simulation	multi-Gaussian	Continuous
Truncated Gaussian kriging	Truncated Gaussian simulation	multi-Gaussian	Categorical
Plurigaussian kriging	Plurigaussian simulation	multi-Gaussian	Categorical

The conditional distributions of the underlying GRF in conjunction with the truncation rule can be used as input to p-field simulation [9] in order to generate realisations of the lithofacies that honour the contact relationships and lithofacies indicator variograms.

CONCLUSIONS

A methodology to obtain the probabilities of occurrence of lithofacies and to calculate the most probable lithofacies has been presented. It allows generating lithofacies maps in a more geological way by considering and reproducing the contact relationships between lithofacies. The proposed approach can be used prior to simulation or as an alternative to the traditionally used indicator kriging. It is extensible to more complex contact relationships by considering two or more GRFs, as done in plurigaussian simulation.

The formulation of the method is robust from a theoretical point of view, since it is based on two well accepted approaches (truncated Gaussian simulation and multi-Gaussian kriging). There is no order relation violation or border effect. The non-stationary case can be addressed by a procedure similar to the one used in plurigaussian simulation, by incorporating local proportion curves.

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REFERENCES

Journel, A.G. (1983) *Nonparametric Estimation of Spatial Distributions*. Mathematical Geology 15(3), pp. 445-468. [1]

Alabert, F. (1987) *Stochastic Imaging of Spatial Distributions Using Hard and Soft Information*. Master's Thesis, Stanford University, Department of Applied Earth Sciences, 197 pp. [2]

Emery, X. (2004) *Properties and Limitations of Sequential Indicator Simulation*. Stochastic Environmental Research and Risk Assessment 18(6), pp. 414-424. [3]

Matheron, G., Beucher, H., de Fouquet, C., Galli, A., Guérillot, D., & Ravenne, C. (1987) *Conditional Simulation of the Geometry of Fluvio Deltaic Reservoirs*. In: 62nd Annual Technical Conference and Exhibition of the Society of Petroleum Engineers, Dallas, Texas. SPE 16753, pp. 123-131. [4]

Geman, S., & Geman, D. (1984) *Stochastic Relaxation, Gibbs Distribution and the Bayesian Restoration of Images*. I.E.E.E. Transactions on Pattern Analysis and Machine Intelligence 6(6), pp. 721-741. [5]

Galli, A., Beucher, H., Le Loc'h, G., Doligez, B., & Heresim Group (1994) *The Pros and Cons of the Truncated Gaussian Method*. In: Armstrong, M. and Dowd, P.A., eds., Geostatistical Simulations. Kluwer Academic, Dordrecht, pp. 217-233. [6]

Armstrong, M., Galli, A., Le Loc'h, G., Geffroy, F., & Eschard, R. (2003) *Plurigaussian Simulations in Geosciences*. Springer, Berlin, 160 pp. [7]

Verly, G. (1983) *The Multigaussian Approach and its Applications to the Estimation of Local Reserves*. Mathematical Geology 15(2), pp. 259-286. [8]

Srivastava, R.M. (1992) *Reservoir Characterization with Probability Field Simulation*. In: 67th Annual Technical Conference and Exhibition of the Society of Petroleum Engineers, Washington. SPE 24753. SPE Formation Evaluation 7(4), pp. 927-937. [9]